On the Achievable Rate-Regions for State-Dependent Gaussian Interference Channel

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Abstract

In this paper, we study a general additive state-dependent Gaussian interference channel (ASD-GIC) where we consider two-user interference channel with two independent states known non-causally at both transmitters, but unknown to either of the receivers. An special case, where the additive states over the two links are the same is studied in [1], [2], in which it is shown that the gap between the achievable symmetric rate and the upper bound is less than $\frac{1}{4}$ bit for the strong interference case. Here, we also consider the case where interference link gains satisfy $a_{12} \geq \frac{N_1}{N_2}$ and $a_{21} \geq \frac{N_2}{N_1}$ (N_i is the channel noise variance) and each channel state has unbounded variance [3], which is referred to as the strong interferences. We first obtain an outer bound on the capacity region. By utilizing lattice-based coding schemes, we obtain four achievable rate regions. Depend on noise variance and channel power constraint, achievable rate regions can coincide with the channel capacity region. For the symmetric model, the achievable sum-rate reaches to within 0.661 bit of the channel capacity for signal to noise ratio (SNR) greater than one.

I. Introduction

An interference channel (IC) can be seen as a model for single-hop multiple one-to-one communications, such as pairs of base stations transmitting over a frequency band that suffers from intercell interference. The earliest research on IC was initiated by Shannon [4]. Unfortunately, the problem of characterizing the capacity region of the general IC has been open for more than 30 years. Except for very strong Gaussian IC, strong Gaussian IC and the sum-capacity for the degraded Gaussian IC, characterizing the capacity region of a Gaussian IC is still an open problem [5], [6], [7]. By utilizing the superposition coding scheme, Carleial obtains an achievable rate region [8]. The best achievable rate region known to date for a Gaussian IC, based on applying rate splitting at the transmitters and simultaneous decoding at the receivers, is established by Han and Kobayashi [9]. Etkin et.al, by deriving new outer bounds, show that an explicit Han-Kobayashi version scheme can achieve capacity region within 1 bits for all channel parameters [10].

Many versions of the IC have also been studied in the literature, including the IC with partial transmitter cooperation [11], the IC with conferencing encoders/decoders [12], [13], the Gaussian IC with feedback [14] and the Gaussian IC with potent relay [15]. In [1], the two-user state-dependent symmetric Gaussian interference channel, where the additive states over the two links are the same, is studied, in which it is shown that the gap between the achievable symmetric rate and the upper bound is less than $\frac{1}{4}$ bit for the strong interference case and less than $\frac{3}{4}$ bit for the weak interference case. In [16] an active interference cancellation mechanism, which is a generalized dirty-paper coding technique, to partially eliminate the effect of the state at the receivers is investigated. It is shown that active interference cancellation significantly enlarges the achievable rate-region.

In this paper, we study another type of the Gaussian IC: the state-dependent two-user IC with two independent states known non-causally at both transmitters, but unknown to either of the receivers. This situation may arise in a multi-cell downlink communication scenario, where two interested cells are interfering with each other and the mobiles suffer from some independent interference (which can be from other neighboring cells' base-stations and considered as an state) non-causally known at each of the base-stations. In addition, we consider the interferences as arbitrary, or equivalently Gaussians with unbounded variances, and channel gains are larger than one (in the symmetric model), which is refereed to as the strong interference [3]. We provide an achievable rate-region based on lattice codes.

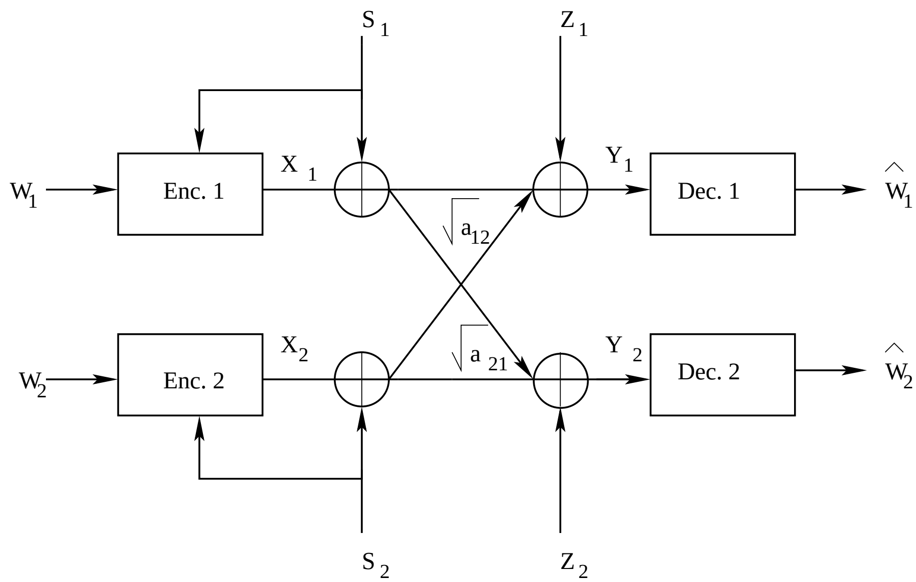

Figure 1. The Gaussian interference channel with common interference known non-causally at both encoders.

A comprehensive study on the performance of lattices is presented in [17]. The problem of achieving an additive white Gaussian noise (AWGN) channel capacity by utilizing lattice codes is studied in [18]. A dirty paper AWGN channel in which the interference is known non-causally or causally at the transmitter is investigated in [19]. For the non-causal case, it is proved that the capacity loss due to applying the lattice strategy for Gaussian noise is upper-bounded by $\frac{1}{2}\log\left(2\pi e G\left(\Lambda\right)\right)$, where $G\left(\Lambda\right)$ is the normalized second moment of the lattice. If the lattice code satisfies the following condition, $\lim\limits_{n\to\infty} G\left(\Lambda\right) = \frac{1}{2\pi e}$, this result coincides with the results of Costa's dirty-paper coding (DPC) [20]. In [3], it is shown that the lattice coding strategy may outperform the DPC in doubly dirty multiple-access channel (MAC). By establishing an outer bound for doubly dirty MAC, Wang is proved that the achievable rate-region by layered lattice scheme is within a constant gap, which is independent of all channel parameters, from the capacity region [21]. In [22], we also show that if the noise variance satisfy a constraint, then the capacity region of an ASD-GIC with common channel state is achieved when the state power goes to infinity.

In this work, we use a lattice-based coding scheme to obtain four achievable rate regions for ASD-GIC. By comparing with an outer bound, which is established for an asymptotic case, where the channel state is assumed to be Gaussian with unbounded variance, we evaluate each achievable rate region. We observe that the lattice based coding scheme can achieve the capacity of the channel under some conditions, which depend on the noise variance and power constraint of the channel. For symmetric ASD-GIC, the achievable rate region of the lattice-based scheme, dependent on the noise variance, is the capacity region or is within 0.661 bit of the channel capacity for signal to noise ratio (SNR) larger than one.

The remainder of the paper is organized as follows: We present the channel model in Section II. In Section III, an outer bound on the capacity region is obtained. Lattice-based achievable rate-regions are presented in Section IV-B. Section V concludes the paper.

II. Channel Model

A. Notations and Channel Model

Throughout the paper, random variables and their realizations are denoted by capital and small letters, respectively. $\boldsymbol{x}$ stands for a vector of length n, $(x_1, x_2, ..., x_n)$. Also, $\|.\|$ denotes the Euclidean norm, and all logarithms are with respect to base 2.

In this paper, an additive state-dependent Gaussian interference channel (ASD-GIC) where the channel states information are independent and known non-causally at both encoders is considered. The system model is depicted in Fig. 1. This channel can

be described by the following following equations (after suitable normalization):

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1, \\ \boldsymbol{Y}_2 &= \sqrt{a_{21}}\boldsymbol{X}_1 + \boldsymbol{X}_2 + \sqrt{a_{21}}\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{Z}_2, \end{aligned}$$

where $\boldsymbol{X}_i$ is the channel input, $\boldsymbol{S}_i$ is an additive arbitrary distributed interference with variance Q_i (or equivalently Gaussian with variance going to infinity), and $\boldsymbol{Z}_i$ represents an AWGN of mean zero and variance N_i. In this work, we consider the strong Gaussian IC with state information, i.e., the interference link gains satisfy $a_{12} \geq \frac{N_1}{N_2}$ and $a_{21} \geq \frac{N_2}{N_1}$ [2] and each channel state has unbounded variance [3].

The message W_i at each encoder is mapped to $\boldsymbol{X}_i$ based on the non-causally known state information $\boldsymbol{S}_i$. Note that $|\mathcal{W}_1| = 2^{nR_1}$ and $|\mathcal{W}_2| = 2^{nR_2}$. Transmitted sequences $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are average-power limited to $P_i > 0$, i.e.,

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left[|X_i[j]|^2\right] \leq P_i, \quad \text{for } i = 1, 2. \tag{1}$$

Each receiver needs to decode the information from the intended transmitter. Based on the channel output, $\boldsymbol{Y}_i$, each receiver makes an estimate of the corresponding message W_i as $\hat{W}_1$. The average error probability is defined as:

$$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2)}}\sum_{w_1,w_2}\Pr\left\{\hat{W}_1 \neq W_1 \, or \, \hat{W}_2 \neq W_2 | (W_1, W_2) \text{ is sent}\right\},$$

where (W_1, W_2) is assumed to be uniformly distributed over $\left\{1, 2, ..., 2^{nR_1}\right\} \times \left\{1, 2, ..., 2^{nR_2}\right\}$. A rate pair (R_1, R_2) is achievable if there exist a sequence of length-n code $C^n(R_1, R_2)$ such that $P_e^{(n)} \to 0$ as $n \to \infty$ [23].

In the following, we present an outer bound over the capacity region for $Q_i \to \infty$.

III. Outer bound

To obtain an outer bound, we use the similar approach as [5]. First, we assume that $\boldsymbol{S}_2$ is known at both decoders. Thus, we can consider the following channel model:

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \boldsymbol{Z}_1, \\ \boldsymbol{Y}_2 &= \sqrt{a_{21}}\boldsymbol{X}_1 + \boldsymbol{X}_2 + \sqrt{a_{21}}\boldsymbol{S}_1 + \boldsymbol{Z}_2. \end{aligned} \tag{2}$$

Now, by dividing $\boldsymbol{Y}_2$ over $\sqrt{a_{21}}$, we get

$$\boldsymbol{Y}_2^{'} \triangleq \frac{\boldsymbol{Y}_2}{\sqrt{a_{21}}} = \boldsymbol{X}_1 + \frac{\boldsymbol{X}_2}{\sqrt{a_{21}}} + \boldsymbol{S}_1 + \frac{\boldsymbol{Z}_2}{\sqrt{a_{21}}}.$$

Using Fano's inequality, we know

$$h\left(W_2|\mathbf{Y}_2\right) \leq n\epsilon_n, \tag{3}$$

where $\epsilon_n \to 0$ as $n \to \infty$. By using W_2 and $\boldsymbol{S}_2$, we can construct $\boldsymbol{X}_2$. Similar to Sato's approach, we also construct noise $\boldsymbol{Z}_2^{'} \sim \mathcal{N}\left(0, N_1 - \frac{N_2}{a_{21}}\right)$, which is independent of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$. Since $a_{21} \geq \frac{N_2}{N_1}$, we add $\left(\sqrt{a_{12}} - \frac{1}{\sqrt{a_{21}}}\right)\boldsymbol{X}_2$ and $\boldsymbol{Z}_2^{'}$ to $\boldsymbol{Y}_2^{'}$. Thus, we have

$$\begin{aligned} \boldsymbol{Y}_2^{''} &\triangleq \boldsymbol{Y}_2^{'} + \left(\sqrt{a_{12}} - \frac{1}{\sqrt{a_{21}}}\right)\boldsymbol{X}_2 + \boldsymbol{Z}_2^{'} \\ &= \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \boldsymbol{Z}_2^{'} + \frac{\boldsymbol{Z}_2}{\sqrt{a_{21}}} \end{aligned}$$

$$= \quad \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \boldsymbol{Z}_1^{'}, \tag{4}$$

where $\boldsymbol{Z}_1^{'} \sim \mathcal{N}(0, N_1)$. Therefore, by comparing (2) and (4), we have

$$h(W_1|\boldsymbol{Y}_2) \leq n\epsilon_n. \tag{5}$$

Thus, from (3) and (5), we get

$$h(W_1, W_2|\boldsymbol{Y}_2) \leq n\epsilon_n. \tag{6}$$

Now, by considering the above-mentioned model, we obtain an outer bound over the ASD-GIC capacity region.

Theorem 1. *In the limit of strong Gaussian interferences, the capacity region of the ASD-GIC is contained in the following region:*

$$R_1 + R_2 \leq \min\left(\frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right), \frac{1}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right)\right). \tag{7}$$

Proof: We bound the sum rate $R_1 + R_2$ as

$$\begin{aligned} n(R_1+R_2) &\leq h(W_1, W_2), \\ &= h(W_1, W_2|\boldsymbol{Y}_2) + I(W_1, W_2; \boldsymbol{Y}_2), \\ &\leq n\epsilon_n + I(W_1, W_2; \boldsymbol{Y}_2), \end{aligned} \tag{8}$$

where (8) follows from (6). Now, we assume that $\boldsymbol{S}_2$ is known at both decoders. Then, we have

$$\begin{aligned} I(W_1, W_2; \boldsymbol{Y}_2) &= h(\boldsymbol{Y}_2) - h(\boldsymbol{Y}_2|W_1, W_2), \\ &\leq h(\boldsymbol{Y}_2) - h(\boldsymbol{Y}_2|\boldsymbol{X}_2, W_1, W_2), && (9) \\ &= h(\boldsymbol{Y}_2) - h(\boldsymbol{Y}_2|\boldsymbol{S}_1, W_1, W_2, \boldsymbol{X}_2) - I(\boldsymbol{S}_1; \boldsymbol{Y}_2|W_1, W_2, \boldsymbol{X}_2), \\ &= h(\boldsymbol{Y}_2) - h(\boldsymbol{Z}_2) - h(\boldsymbol{S}_1|W_1, W_2, \boldsymbol{X}_2) + h(\boldsymbol{S}_1|W_1, W_2, \boldsymbol{X}_2, \boldsymbol{Y}_2), \\ &\leq h(\boldsymbol{Y}_2) - h(\boldsymbol{Z}_2) - h(\boldsymbol{S}_1) + h(\boldsymbol{X}_1 + \frac{\boldsymbol{Z}_2}{\sqrt{a_{21}}}), && (10) \\ &\leq \frac{n}{2}\log\left(\frac{N_2 + \left(\sqrt{P_2} + \sqrt{a_{21}P_1} + \sqrt{a_{21}Q_1}\right)^2}{a_{21}Q_1}\right) + \frac{n}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right), && (11) \end{aligned}$$

where (9) follows from the fact that conditioning reduces entropy, (10) follows from the fact that $\boldsymbol{S}_1$ is independent to $(W_1, W_2, \boldsymbol{X}_2)$ and (11) follows from the fact that Gaussian distribution maximizes differential entropy for a fixed second moment and Cauchy-Schwarz inequality. In the limit of strong interference, i.e., $Q_1 \to +\infty$, we get

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right). \tag{12}$$

Similarly, by assuming $\boldsymbol{S}_1$ is known at both decoders, and reforming the above equations, we get

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right). \tag{13}$$

By combining (12) and (13), we obtain

$$R_1 + R_2 \leq \min\left(\frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right), \frac{1}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right)\right). \tag{14}$$

■

IV. Achievable Rate-Regions

A. An Achievable Rate-Region based on Random Binning Scheme

An achievable region for this channel can be obtained by random binning argument. It is known that for the traditional strong Gaussian IC, the capacity region is simply the intersection of two MAC rate-regions [5]. Based on this idea, we can consider ASD-GIC as two doubly-dirty MACs. The best rate-region for a doubly dirty MAC, based on the random binning technique, is presented in [24], that is given by the convex hull of all rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &\leq I\left(U_1;Y|U_2\right) - I(U_1;S_1), \\ R_2 &\leq I\left(U_2;Y|U_1\right) - I(U_2;S_2), \\ R_1+R_2 &\leq I\left(U_1,U_2;Y\right) - I(U_1;S_1) - I(U_2;S_2), \end{aligned} \tag{15}$$

for some $p(u_1,u_2,x_1,x_2|s_1,s_2) = p(u_1,x_1|s_1)p(u_2,x_2|s_2)$. Now, for $Q_i \to \infty$, we can evaluate the achievable sum rate in (15) for both MACs and show that we cannot achieve any positive rates using such random binning scheme. This evaluation is quite similar to the presented approach in [3] and is provided in Appendix.

B. Lattice Alignment

1) Lattice Definitions: Here, we provide some necessary definitions on lattices and nested lattice codes [18], [25], [26].

An *n-dimensional lattice* Λ is a set of points in Euclidean space $\mathbb{R}^n$ such that, if $\boldsymbol{x},\boldsymbol{y} \in \Lambda$, then $\boldsymbol{x}+\boldsymbol{y} \in \Lambda$, and if $\boldsymbol{x} \in \Lambda$, then $-\boldsymbol{x} \in \Lambda$. A lattice Λ can always be written in terms of a generator matrix $\mathbf{G} \in \mathbb{Z}^{n\times n}$ as

$$\Lambda = \{\boldsymbol{x} = \boldsymbol{z}\mathbf{G} : \boldsymbol{z} \in \mathbb{Z}^n\},$$

where $\mathbb{Z}$ represents integers.

The *nearest neighbor quantizer* $\mathcal{Q}(.)$ associated with lattice Λ is

$$\mathcal{Q}_\Lambda(\boldsymbol{x}) = \arg\min_{\boldsymbol{l}\in\Lambda} \|\boldsymbol{x} - \boldsymbol{l}\|.$$

The *fundamental Voronoi region* of lattice Λ is set of points in $\mathbb{R}^n$ closest to the zero codeword, i.e.,

$$\mathcal{V}_0(\Lambda) = \{\boldsymbol{x} \in \mathbb{R}^n : \mathcal{Q}(\boldsymbol{x}) = 0\}.$$

$\sigma^2\left(\Lambda\right)$ which is called the second moment of lattice Λ is defined as

$$\sigma^2(\Lambda) = \frac{1}{n}\frac{\int_{\mathcal{V}(\Lambda)} \|\boldsymbol{x}\|^2 \, d\boldsymbol{x}}{\int_{\mathcal{V}(\Lambda)} d\boldsymbol{x}}, \tag{16}$$

and the *normalized second moment* of lattice Λ can be expressed as

$$G(\Lambda) = \frac{\sigma^2(\Lambda)}{[\int_{\mathcal{V}(\Lambda)} d\boldsymbol{x}]^{\frac{2}{n}}} = \frac{\sigma^2(\Lambda)}{V^{\frac{2}{n}}},$$

where $V = \int_{\mathcal{V}(\Lambda)} d\boldsymbol{x}$ is the Voronoi region volume, i.e., $V = \text{Vol}(\mathcal{V})$.

The *modulo-Λ operation* with respect to lattice Λ is defined as

$$\boldsymbol{x} \text{ mod } \Lambda = \boldsymbol{x} - \mathcal{Q}(\boldsymbol{x}),$$

that maps $\boldsymbol{x}$ into a point in the fundamental Voronoi region. The modulo lattice operation satisfies the following distributive property

$$[\boldsymbol{x} \text{ mod } \Lambda + \boldsymbol{y}] \text{ mod } \Lambda = [\boldsymbol{x} + \boldsymbol{y}] \text{ mod } \Lambda.$$

A sequence of lattices $\Lambda^{(n)} \subseteq \mathbb{R}^n$ is good for mean-squared error (MSE) quantization if

$$\lim_{n\to\infty} G\left(\Lambda^{(n)}\right) = \frac{1}{2\pi e}.$$

The sequence is indexed by the lattice dimension n. The existence of such lattices is shown in [27], [28].

Let $\boldsymbol{Z}$ be a length-$n i.i.d$ Gaussian vector, $\boldsymbol{Z} \sim \mathcal{N}\left(\mathbf{0}, \sigma_Z^2 \boldsymbol{I}_n\right)$. The volume-to-noise ratio of a lattice is given by

$$\mu\left(\Lambda, \epsilon\right) = \frac{\left(\text{Vol}(\mathcal{V})\right)^{2/n}}{\sigma_Z^2},$$

where σ_Z^2 is chosen such that $\Pr\left\{\boldsymbol{Z} \notin \mathcal{V}\right\} = \epsilon$ and $\boldsymbol{I}_n$ is an $n \times n$ identity matrix. A sequence of lattices $\Lambda^{(n)}$ is Poltyrev-good if

$$\lim_{n\to\infty} \mu\left(\Lambda^{(n)}, \epsilon\right) = 2\pi e, \quad \forall \epsilon \in (0,1)$$

and for fixed volume-to-noise ratio, greater than $2\pi e$, $\Pr\left\{\boldsymbol{Z} \notin \mathcal{V}^n\right\}$ decays exponentially in n . Poltyrev showed that a sequence of such lattices exists [29]. The existence of a sequence of lattices $\Lambda^{(n)}$ which are good in both senses (i.e., simultaneously are Poltyrev-good and Rogers-good) is shown in [28].

We now calculate differential entropy of an n-dimensional random vector $\boldsymbol{D}$ which is distributed uniformly over fundamental Voronoi region. We have [27]

$$\begin{aligned} h\left(\boldsymbol{D}\right) &= \log\left(V\right), \\ &= \log\left(\frac{\sigma^2(\Lambda)}{G\left(\Lambda\right)}\right)^{n/2}, \\ &\approx \frac{n}{2}\log\left(2\pi e \sigma^2(\Lambda)\right), \end{aligned} \tag{17}$$

where the last approximation holds for lattices which are good for quantization.

In the following, we present a key property of dithered lattice codes.

Lemma 2. ***The Crypto Lemma [30], [18]*** *Let $\mathbf{V}$ be a random vector with an arbitrary distribution over $\mathbb{R}^n$. If $\boldsymbol{D}$ is independent of $\mathbf{V}$ and uniformly distributed over $\mathcal{V}$, then $(\mathbf{V} + \boldsymbol{D})$ mod Λ is also independent of $\mathbf{V}$ and uniformly distributed over $\mathcal{V}$.*

Proof: See Lemma 2 in ***[30].*** ■

2) Imbalanced ASD-GIC:

Theorem 3. *Imbalanced SNRs: Suppose that $N_1 \leq \sqrt{a_{12}P_2P_1} - a_{12}P_2$ and $N_2 \leq \sqrt{a_{21}P_2P_1} - a_{21}P_1$. The capacity region of an ASD-GIC in the limit of strong Gaussian interferences, i.e., $\boldsymbol{S}_i \sim \mathcal{N}(0, Q_i)$ and $Q_i \to +\infty$, is given by:*

$$R_1 + R_2 \quad \leq \quad \min\left(\frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right), \frac{1}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right)\right).$$

Proof: Based on the outer region in Section III, the converse is proved. Here, we show achievability of the following region using a lattice-based coding scheme:

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right),$$

where $N_1 \leq \sqrt{a_{12}P_2P_1} - a_{12}P_2$. Suppose that there exist three lattices Λ_1, Λ_2 and $\Lambda_3 = \sqrt{a_{12}}\Lambda_2$, which are good for quantization ($\lim_{n\to\infty} G\left(\Lambda_i\right) = \frac{1}{2\pi e}$, for $i = 1, 2, 3$), such that

$$\sigma^2\left(\Lambda_1\right) = P_1,\ \sigma^2\left(\Lambda_2\right) = P_2,\ \text{ and } \sigma^2\left(\Lambda_3\right) = a_{12}P_2.$$

User 1 and user 2 use lattices Λ_1 and Λ_2 with second moments P_1 and P_2, respectively. It is also assumed that $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are two independent dithers, where $\boldsymbol{D}_1$ is uniformly distributed over the Voronoi region $\mathcal{V}_1$ and $\boldsymbol{D}_2$ is uniformly distributed

over the Voronoi region $\mathcal{V}_2$. Dithers are known at the decoders.

First, we achieve the following corner point

$$(R_1, R_2) = \left(0, \frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right)\right),$$

where $a_{12}P_2\left(\frac{a_{12}P_2+N_1}{a_{12}P_2}\right)^2 \leq P_1$. We assume that $\Lambda_3 = \alpha\Lambda_1$. The encoders send

$$\begin{aligned}\boldsymbol{X}_1 &= [-\boldsymbol{S}_1 - \boldsymbol{D}_1] \mod \Lambda_1,\\ \boldsymbol{X}_2 &= [\boldsymbol{V}_2 - \alpha\boldsymbol{S}_2] \mod \Lambda_2,\end{aligned}$$

where $\boldsymbol{V}_2 \sim$ Unif$\left(\mathcal{V}_2\right)$ carry the information for user 2, and dither $\boldsymbol{D}_1$ is known at the encoder of user 1. At the receiver of user 1, based on the channel output, given by

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1.$$

the following operation is performed:

$$\begin{aligned}\boldsymbol{Y}_{d1} &= [\alpha\boldsymbol{Y}_1 + \alpha\boldsymbol{D}_1] \mod \Lambda_3,\\ &= [\alpha\left([-\boldsymbol{S}_1 - \boldsymbol{D}_1] \mod \Lambda_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1\right) + \alpha\boldsymbol{D}_1] \mod \Lambda_3,\\ &= [\sqrt{a_{12}}\boldsymbol{V}_2 + \alpha\sqrt{a_{12}}\boldsymbol{X}_2 - \sqrt{a_{12}}\left(\boldsymbol{V}_2 - \alpha\boldsymbol{S}_2\right) + \alpha\boldsymbol{Z}_1 - \alpha\mathcal{Q}_{\Lambda_1}(-\boldsymbol{S}_1 - \boldsymbol{D}_1)] \mod \Lambda_3,\\ &= [\sqrt{a_{12}}\boldsymbol{V}_2 + (\alpha - 1)\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha\boldsymbol{Z}_1] \mod \Lambda_3, \qquad (18)\\ &= [\sqrt{a_{12}}\boldsymbol{V}_2 + \boldsymbol{Z}_{eff}] \mod \Lambda_3,\end{aligned}$$

where

$$\boldsymbol{Z}_{eff} = [(\alpha - 1)\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha\boldsymbol{Z}_1] \mod \Lambda_3.$$

(18) follows from $\Lambda_3 = \alpha\Lambda_1$, thus we have $\alpha\mathcal{Q}_{\Lambda_1}(-\boldsymbol{S}_1 - \boldsymbol{D}_1)) \in \Lambda_3$, i.e., the interference signal is aligned with Λ_3. Hence, this term disappears after the modulo operation. To calculate rate R_2, it is assumed that $\boldsymbol{V}_2 \sim$ Unif$\left(\mathcal{V}_2\right)$. We have

$$\begin{aligned}R_2 &= \frac{1}{n}I\left(\boldsymbol{V}_2; \boldsymbol{Y}_{d1}\right),\\ &= \frac{1}{n}\left\{h(\boldsymbol{Y}_{d1}) - h(\boldsymbol{Y}_{d1}|\boldsymbol{V}_2)\right\}\\ &= \frac{1}{2}\log\left(\frac{\sigma^2\left(\Lambda_3\right)}{G\left(\Lambda_3\right)}\right) - \frac{1}{n}h\left([(\alpha - 1)\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha\boldsymbol{Z}_1] \mod \Lambda_3\right), \qquad (19)\\ &\geq \frac{1}{2}\log\left(\frac{a_{12}P_2}{a_{12}\left(\alpha - 1\right)^2 P_2 + \alpha^2 N_1}\right) - \frac{1}{2}\log\left(2\pi e G\left(\Lambda_3\right)\right), \qquad (20)\end{aligned}$$

where (19) follows from the fact that $\sqrt{a_{12}}\boldsymbol{V}_2$ is uniform over $\mathcal{V}_3 = \sqrt{a_{12}}\mathcal{V}_2$; thus $\boldsymbol{Y}_{d1}$ is also uniformly distributed over $\mathcal{V}_3$ (crypto lemma) and then we can apply (17). (20) follows from the fact that modulo operation reduces the second moment and entropy is maximized by the normal distribution for a fixed second moment. Now, we need to find the coefficient α such that minimizes the mean squared error (MSE) of the effective noise $\boldsymbol{Z}_{\text{eff}}$. Hence,

$$\alpha_{\text{MMSE}} = \frac{a_{12}P_2}{a_{12}P_2 + N_1}.$$

Applying the optimal α and a good quantization lattice Λ_1, we can achieve the following corner point

$$(R_1, R_2) = \left(0, \frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right)\right). \qquad (21)$$

Clearly, for $a_{12}P_2\left(\frac{a_{12}P_2+N_1}{a_{12}P_2}\right)^2 = P_1$ the inner bound meets the outer bound (14). Also, for $a_{12}P_2\left(\frac{a_{12}P_2+N_1}{a_{12}P_2}\right)^2 \le P_1$, the outer bound remains $\frac{1}{2}\log\left(1+\frac{a_{12}P_2}{N_1}\right)$, thus the outer bound is also achievable.

Now, we achieve the following corner point

$$(R_1, R_2) = \left(\frac{1}{2}\log\left(1+\frac{a_{12}P_2}{N_1}\right), 0\right),$$

where $a_{12}P_2\left(\frac{a_{12}P_2+N_1}{a_{12}P_2}\right)^2 \le P_1$. We assume that $\Lambda_3 = \alpha\Lambda_1$. The encoders send

$$\begin{aligned} \boldsymbol{X}_1 &= [\boldsymbol{V}_1 - \boldsymbol{S}_1] \mod \Lambda_1, \\ \boldsymbol{X}_2 &= [-\alpha\boldsymbol{S}_2 - \boldsymbol{D}_2] \mod \Lambda_2. \end{aligned}$$

where $\boldsymbol{V}_1 \sim \text{ Unif}\,(\mathcal{V}_1)$ carry the information for user 1, the dither $\boldsymbol{D}_2$ is known at the encoder of user 2. The signal at receiver 1 is given by

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1.$$

At the receiver, the following operation is performed:

$$\begin{aligned} \boldsymbol{Y}_{d1} &= [\alpha\boldsymbol{Y}_1 + \sqrt{a_{12}}\boldsymbol{D}_2] \mod \Lambda_3, \\ &= [\alpha\left([\boldsymbol{V}_1 - \boldsymbol{S}_1] \mod \Lambda_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1\right) + \sqrt{a_{12}}\boldsymbol{D}_2] \mod \Lambda_3, \\ &= [\alpha\boldsymbol{V}_1 + \alpha\sqrt{a_{12}}\boldsymbol{X}_2 - \sqrt{a_{12}}\left(-\alpha\boldsymbol{S}_2 - \boldsymbol{D}_2\right) + \alpha\boldsymbol{Z}_1 - \alpha\mathcal{Q}_{\Lambda_1}(\boldsymbol{V}_1 - \boldsymbol{S}_1)] \mod \Lambda_3, \\ &= [\alpha\boldsymbol{V}_1 + (\alpha-1)\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha\boldsymbol{Z}_1] \mod \Lambda_3, \qquad (22) \\ &= [\alpha\boldsymbol{V}_1 + \boldsymbol{Z}_{eff}] \mod \Lambda_3, \end{aligned}$$

where

$$Z_{eff} = [(\alpha-1)\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha\boldsymbol{Z}_1] \mod \Lambda_3,$$

and (22) is based on $\alpha\Lambda_1 = \Lambda_3$. To calculate the rate R_2, it is assumed that $\boldsymbol{V}_1 \sim \text{ Unif}\,(\mathcal{V}_1)$. We have

$$\begin{aligned} R_1 &= \frac{1}{n}I\left(\boldsymbol{V}_1; \boldsymbol{Y}_{d1}\right), \\ &= \frac{1}{n}\left\{h(\boldsymbol{Y}_{d1}) - h(\boldsymbol{Y}_{d1}|\boldsymbol{V}_1)\right\}, \\ &= \frac{1}{2}\log\left(\frac{a_{12}P_2}{G\left(\Lambda_3\right)}\right) - \frac{1}{n}h\left([(\alpha-1)\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha\boldsymbol{Z}_1] \mod \Lambda_3\right), \qquad (23) \\ &\ge \frac{1}{2}\log\left(\frac{a_{12}P_2}{(\alpha-1)^2 a_{12}P_2 + \alpha^2 N_1}\right) - \frac{1}{2}\log\left(2\pi e G\left(\Lambda_3\right)\right), \qquad (24) \end{aligned}$$

where (23) follows from $\alpha^2 P_1 = a_{12}P_2$ and the fact that $\alpha\boldsymbol{V}_1$ is uniformly distributed over $\mathcal{V}_3$; thus $\boldsymbol{Y}_{d1}$ is also uniform over $\mathcal{V}_3$ (crypto lemma). (24) is based on this fact that the second moment is increased by removing modulo, and also for a fixed second moment, Gaussian distribution maximizes differential entropy. By considering MMSE coefficient that minimizes the MSE of the effective noise $\boldsymbol{Z}_{\text{eff}}$, i.e., $\alpha = \frac{a_{12}P_2}{a_{12}P_2+N_1}$ and applying a good quantization lattice Λ_3, we can achieve the following corner point

$$(R_1, R_2) = \left(\frac{1}{2}\log\left(1+\frac{a_{12}P_2}{N_1}\right), 0\right). \qquad (25)$$

Clearly, for $a_{12}P_2\left(\frac{a_{12}P_2+N_1}{a_{12}P_2}\right)^2 = P_1$ the inner bound meets the outer bound (14). Likewise, for $a_{12}P_2\left(\frac{a_{12}P_2+N_1}{a_{12}P_2}\right)^2 \le P_1$, the outer bound remains $\frac{1}{2}\log\left(1+\frac{a_{12}P_2}{N_1}\right)$, thus the outer bound is also achievable.

By using time sharing between two corner points, (21) and (25), for decoder 1, we can achieve the following sum-rate region:

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1+\frac{a_{12}P_2}{N_1}\right). \tag{26}$$

If $N_2 \leq \sqrt{a_{12}P_2P_1} - a_{12}P_1$, by similar analysis at decoder 2, we have

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1+\frac{a_{21}P_1}{N_2}\right). \tag{27}$$

Therefore by using (26) and (27), we get the following achievable rate region for ASD-GIC:

$$R_1 + R_2 \quad \leq \quad \min\left(\frac{1}{2}\log\left(1+\frac{a_{12}P_2}{N_1}\right), \frac{1}{2}\log\left(1+\frac{a_{21}P_1}{N_2}\right)\right).$$

■

3) Nearly Balanced ASD-GIC:

Theorem 4. *If $N_1 \geq \sqrt{a_{12}P_2P_1} - \min(a_{12}P_2, P_1)$ and $N_2 \geq \sqrt{a_{21}P_2P_1} - \min(a_{21}P_1, P_2)$ for $P_1 \neq a_{12}P_2$ and $a_{21}P_1 \neq P_2$, then, the following region is achievable for ASD-GIC:*

$$R_1+R_2 \leq \min\left(u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_1+a_{12}P_2+N_1}{2N_1+\left(\sqrt{P_1}-\sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right\}, u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_2+a_{21}P_1+N_2}{2N_2+\left(\sqrt{P_2}-\sqrt{a_{21}P_1}\right)^2}\right)\right]^+\right\}\right), \tag{28}$$

where the upper convex envelope is with respect to P_1 and P_2.

Proof: We use the lattice-based coding scheme. Suppose that there exist three lattices Λ_1, Λ_2 and $\Lambda_3 = \sqrt{a_{12}}\Lambda_2$, which are good for quantization ($\lim_{n\to\infty} G(\Lambda_i) = \frac{1}{2\pi e}$, for $i = 1, 2, 3$), such that

$$\sigma^2(\Lambda_1) = P_1,\ \sigma^2(\Lambda_2) = P_2,\ \text{and}\ \sigma^2(\Lambda_3) = a_{12}P_2.$$

User 1 and user 2 use the lattices Λ_1 and Λ_2 with second moments P_1 and P_2, respectively. It is also assumed that $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are two independent dithers that $\boldsymbol{D}_1$ is uniformly distributed over the Voronoi region $\mathcal{V}_1$ and $\boldsymbol{D}_2$ is uniformly distributed over the Voronoi region $\mathcal{V}_2$. Dithers are known at the decoders.

First, we consider $a_{12}P_2 \leq \frac{(P_1+N_1)^2}{P_1}$ or equivalently $N_1 \geq \sqrt{a_{12}P_1P_2} - P_1$. We assume that $\Lambda_3 = \frac{\alpha_2}{\alpha_1}\Lambda_1$. The encoders send

$$\begin{aligned}
\boldsymbol{X}_1 &= [-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1] \bmod \Lambda_1,\\
\boldsymbol{X}_2 &= [\boldsymbol{V}_2 - \alpha_2\boldsymbol{S}_2 - \boldsymbol{D}_2] \bmod \Lambda_2.
\end{aligned}$$

At the receiver of user 1, based on the channel output given by

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1,$$

the following operation is performed:

$$\begin{aligned}
\boldsymbol{Y}_{d1} &= \left[\alpha_2\boldsymbol{Y}_1 + \sqrt{a_{12}}\boldsymbol{D}_2 - \frac{\alpha_2}{\alpha_1}\boldsymbol{D}_1\right] \bmod \Lambda_3,\\
&= \left[\alpha_2\left([-\alpha_1\boldsymbol{S}_1+\boldsymbol{D}_1] \bmod \Lambda_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1\right) + \sqrt{a_{12}}\boldsymbol{D}_2 - \frac{\alpha_2}{\alpha_1}\boldsymbol{D}_1\right] \bmod \Lambda_3,\\
&= \Big[\sqrt{a_{12}}\boldsymbol{V}_2 + \alpha_2\left(\sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{Z}_1\right) - \sqrt{a_{12}}\left(\boldsymbol{V}_2 - \alpha_2\boldsymbol{S}_2 - \boldsymbol{D}_2\right) - (1-\alpha_1)\frac{\alpha_2}{\alpha_1}(-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1)\\
&\quad -\alpha_2\mathcal{Q}_{\Lambda_1}(-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1)\Big] \bmod \Lambda_3,\\
&= \left[\sqrt{a_{12}}\boldsymbol{V}_2 + \sqrt{a_{12}}(\alpha_2-1)\boldsymbol{X}_2 - (1-\alpha_1)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1 + \alpha_2\boldsymbol{Z}_1 - \frac{\alpha_2}{\alpha_1}\mathcal{Q}_{\Lambda_1}(-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1)\right] \bmod \Lambda_3,
\end{aligned} \tag{29}$$

$$
\begin{aligned}
&= \left[\sqrt{a_{12}}\boldsymbol{V}_2 + \sqrt{a_{12}}\left(\alpha_2 - 1\right)\boldsymbol{X}_2 - \left(1-\alpha_1\right)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1 + \alpha_2\boldsymbol{Z}_1\right] \bmod \Lambda_3, \qquad (30)\\
&= \left[\sqrt{a_{12}}\boldsymbol{V}_2 + \boldsymbol{Z}_{eff}\right] \bmod \Lambda_3,
\end{aligned}
$$

where

$$
\boldsymbol{Z}_{eff} = \left[\sqrt{a_{12}}\left(\alpha_2 - 1\right)\boldsymbol{X}_2 - \left(1-\alpha_1\right)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1 + \alpha_2\boldsymbol{Z}_1\right] \bmod \Lambda_3.
$$

(29) is based on distributive law and (30) follows from $\Lambda_3 = \frac{\alpha_2}{\alpha_1}\Lambda_1$, we have that $\frac{\alpha_2}{\alpha_1}\mathcal{Q}_{\Lambda_1}(-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1) \in \Lambda_3$, i.e., the interference signal is aligned with Λ_3. Hence, the element disappears after the modulo operation. To calculate the rate R_2, it is assumed that $\boldsymbol{V}_2 \sim \text{ Unif}\left(\mathcal{V}_2\right)$. We have

$$
\begin{aligned}
R_2 &= \frac{1}{n} I\left(\boldsymbol{V}_2; \boldsymbol{Y}_{d1}\right),\\
&= \frac{1}{n}\left\{h(\boldsymbol{Y}_{d1}) - h(\boldsymbol{Y}_{d1}|\boldsymbol{V}_2)\right\}\\
&= \frac{1}{2}\log\left(\frac{a_{12}P_2}{G\left(\Lambda_3\right)}\right) - \frac{1}{n}h\left(\left[\sqrt{a_{12}}\left(\alpha_2 - 1\right)\boldsymbol{X}_2 - \left(1-\alpha_1\right)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1 + \alpha_2\boldsymbol{Z}_1\right] \bmod \Lambda_3\right), \qquad (31)\\
&\geq \frac{1}{2}\log\left(\frac{a_{12}P_2}{a_{12}\left(\alpha_2-1\right)^2 P_2 + \left(\left(1-\alpha_1\right)\frac{\alpha_2}{\alpha_1}\right)^2 P_1 + \alpha_2^2 N_1}\right) - \frac{1}{2}\log\left(2\pi e G\left(\Lambda_3\right)\right), \qquad (32)
\end{aligned}
$$

where (31) follows from $\sqrt{a_{12}}\boldsymbol{V}_2$ is uniform over $\sqrt{a_{12}}\mathcal{V}_2$ therefore $\boldsymbol{Y}_{d1}$ is also uniform over $\sqrt{a_{12}}\mathcal{V}_2$ (crypto lemma) and (32) follows from the fact that modulo operation reduces the second moment and for a fixed second moment, Gaussian distribution maximizes differential entropy. Now, by considering $\left(\frac{\alpha_2}{\alpha_1}\right)^2 P_1 = a_{12}P_2$, and MMSE coefficient α_2, such that the MSE of the effective noise $\boldsymbol{Z}_{\text{eff}}$ is minimized when the lattice dimension goes to infinity, we obtain

$$
\alpha_{2,\text{MMSE}} = \frac{\sqrt{a_{12}P_2}\left(\sqrt{P_1} + \sqrt{a_{12}P_2}\right)}{P_1 + a_{12}P_2 + N_1}.
$$

With this chosen for α_2, we get that the following rate is achievable:

$$
R_2 \leq \left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+. \qquad (33)
$$

Now, we consider $P_1 \leq \frac{(a_{12}P_2 + N_1)^2}{a_{12}P_2}$ or equivalently $N_1 \geq \sqrt{a_{12}P_2P_1} - a_{12}P_2$. We assume that $\Lambda_3 = \frac{\alpha_2}{\alpha_1}\Lambda_1$. The encoders send

$$
\begin{aligned}
\boldsymbol{X}_1 &= \left[-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1\right] \bmod \Lambda_1,\\
\boldsymbol{X}_2 &= \left[\boldsymbol{V}_2 - \alpha_2\boldsymbol{S}_2 + \boldsymbol{D}_2\right] \bmod \Lambda_2.
\end{aligned}
$$

At the receiver of user 1, based on the channel output given by

$$
\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1.
$$

The following operation is performed:

$$
\begin{aligned}
\boldsymbol{Y}_{d1} &= \left[\alpha_1\boldsymbol{Y}_1 - \frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{D}_2 - \boldsymbol{D}_1\right] \bmod \Lambda_1,\\
&= \left[\alpha_1\left(\boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1\right) - \frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{D}_2 - \boldsymbol{D}_1\right] \bmod \Lambda_1,\\
&= \left[\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{V}_2 + \alpha_1\left(\boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{Z}_1\right) - \left(-\alpha_1\boldsymbol{S}_1 + \boldsymbol{D}_1\right) - \frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\left(\boldsymbol{V}_2 - \alpha_2\boldsymbol{S}_2 + \boldsymbol{D}_2\right)\right] \bmod \Lambda_1,
\end{aligned}
$$

$$= \left[\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{V}_2 + (\alpha_1 - 1)\boldsymbol{X}_1 - (1-\alpha_2)\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha_1\boldsymbol{Z}_1\right] \bmod \Lambda_1, \tag{34}$$

$$= \left[\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{V}_2 + \boldsymbol{Z}_{eff}\right] \bmod \Lambda_1,$$

where

$$\boldsymbol{Z}_{eff} = \left[(\alpha_1 - 1)\boldsymbol{X}_1 - (1-\alpha_2)\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha_1\boldsymbol{Z}_1\right] \bmod \Lambda_1.$$

(34) is based on distributive law and follows from $\Lambda_1 = \frac{\alpha_1}{\alpha_2}\Lambda_3$, we have that $\frac{\alpha_1}{\alpha_2}\mathcal{Q}_{\Lambda_3}(\boldsymbol{V}_2 - \alpha_2\boldsymbol{S}_1 + \boldsymbol{D}_2) \in \Lambda_1$. Hence, the element disappears after the modulo operation. To calculate the rate R_2, it is assumed that $\boldsymbol{V}_2 \sim Unif(\mathcal{V}_2)$. We have

$$\begin{aligned}
R_2 &= \frac{1}{n}I(\boldsymbol{V}_2;\boldsymbol{Y}_{d1}), \\
&= \frac{1}{n}\left\{h(\boldsymbol{Y}_{d1}) - h(\boldsymbol{Y}_{d1}|\boldsymbol{V}_2)\right\} \\
&= \frac{1}{2}\log\left(\frac{P_1}{G(\Lambda_1)}\right) - \frac{1}{n}h\left(\left[(\alpha_1 - 1)\boldsymbol{X}_1 - (1-\alpha_2)\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\boldsymbol{X}_2 + \alpha_1\boldsymbol{Z}_1\right] \bmod \Lambda_1\right), && (35) \\
&\geq \frac{1}{2}\log\left(\frac{P_1}{(\alpha_1-1)^2 P_1 + \left((1-\alpha_2)\frac{\alpha_1}{\alpha_2}\right)^2 a_{12}P_2 + \alpha_1^2 N_1}\right) - \frac{1}{2}\log\left(2\pi e G(\Lambda_1)\right), && (36)
\end{aligned}$$

where (34) follows from the fact that $\sqrt{a_{12}}\boldsymbol{V}_2$ is uniform over $\mathcal{V}_1 = \sqrt{a_{12}}\mathcal{V}_2$; thus, $\boldsymbol{Y}_{d1}$ is also uniform over $\mathcal{V}_1$ (crypto lemma). Since modulo operation reduces the second moment and for a fixed second moment, the entropy is maximized by Gaussian distribution, (36) is correct. Now, by considering $\left(\frac{\alpha_1}{\alpha_2}\right)^2 a_{12}P_2 = P_1$, we find the optimal α when the lattice dimension goes to infinity such that minimizes the MSE of the effective noise $\boldsymbol{Z}_{\text{eff}}$. Hence,

$$\alpha_{1,\text{MMSE}} = \frac{\sqrt{P_1}\left(\sqrt{P_1} + \sqrt{a_{12}P_2}\right)}{P_1 + a_{12}P_2 + N_1}.$$

With this chosen for α, we get that any rate

$$R_2 \leq \left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+, \tag{37}$$

is achievable. From (33) and (37), we get the following corner point is achievable

$$(R_1, R_2) = \left(0, \left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right), \tag{38}$$

if

$$N_1 \geq \sqrt{a_{12}P_1P_2} - \min\left(a_{12}P_2, P_1\right).$$

By symmetry, it can be shown that for $N_1 \geq \sqrt{a_{12}P_1P_2} - \min\left(a_{12}P_2, P_1\right)$, the following corner point is achievable (see Appendix 2):

$$(R_1, R_2) = \left(\left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+, 0\right). \tag{39}$$

Thus, for decoder 1, we can achieve the following region by time sharing between two corner points, (38) and (39):

$$R_1 + R_2 \leq \left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+, \tag{40}$$

if

$$N_1 \geq \sqrt{a_{12}P_1P_2} - \min\left(a_{12}P_2, P_1\right).$$

By similar analysis for decoder 2, we can achieve the following region

$$R_1 + R_2 \leq \left[\frac{1}{2}\log\left(\frac{a_{21}P_1 + P_2 + N_2}{2N_2 + \left(\sqrt{P_2} - \sqrt{a_{21}P_1}\right)^2}\right)\right]^+, \tag{41}$$

if

$$N_2 \geq \sqrt{a_{21}P_1P_2} - \min\left(a_{21}P_1, P_2\right).$$

The theorem follows from (40) and (41). ∎

4) Calculating the gap: Now, we obtain the gap between the outer bound in (14) and the achievable rate region, given by (28). First, we define the following gap:

$$\begin{aligned}
\xi\left(P_1, P_2, N_1, N_2, a_{12}, a_{21}\right) &= \min\left(\frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right), \frac{1}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right)\right) \\
&- \min\left(u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right\}\right. \\
&\left. u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_2 + a_{21}P_1 + N_2}{2N_2 + \left(\sqrt{P_2} - \sqrt{a_{21}P_1}\right)^2}\right)\right]^+\right\}\right).
\end{aligned} \tag{42}$$

Since, it is difficult to calculate the gap (42) for asymmetric model, thus, we focus over symmetric model, i.e., $P_1 = P_2 \triangleq P$, $N_1 = N_2 \triangleq N$ and $a_{12} = a_{21} \triangleq a$. We have

$$\xi\left(P, N, a\right) = \frac{1}{2}\log\left(1 + \frac{aP}{N}\right) - u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P\left(1+a\right) + N}{2N + P\left(1 - \sqrt{a}\right)^2}\right)\right]^+\right\}, \tag{43}$$

and the condition on noise variance is reduced to

$$N \geq \left(\sqrt{a} - 1\right)P,$$

where $a \geq 1$ (strong interference). Now, we investigate the second term for obtaining its minimum value. We can see that gap is an increasing function of a for $1 \leq a \leq \left(\frac{P+N}{P}\right)^2$. Therefore, its maximum value occurs at $a = \left(\frac{P+N}{P}\right)^2$. Thus, to obtain the gap, we evaluate it for $a = \left(\frac{P+N}{P}\right)^2$. We have

$$\xi\left(P, N, \left(\frac{P+N}{P}\right)^2\right) = \frac{1}{2}\log\left(1 + \frac{\left(P+N\right)^2}{NP}\right) - u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{2P^2 + 3PN + N^2}{2PN + N^2}\right)\right]^+\right\}. \tag{44}$$

Let us define x as $x \triangleq \frac{P}{N}$. Thus,

$$\xi\left(P, N, \left(\frac{P+N}{P}\right)^2\right) = \frac{1}{2}\log\left(1 + \frac{\left(x+1\right)^2}{x}\right) - u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{2x^2 + 3x + 1}{2x + 1}\right)\right]^+\right\} \triangleq \tilde{\xi}\left(x\right). \tag{45}$$

As we can see in Fig. 2, $\tilde{\xi}\left(x\right)$ is a decreasing function of x. Thus, it is maximized at $x \to 0$. Unfortunately, the gap tends to infinity as $x \to 0$. But, as we can see in Table I, the gap is smaller than 0.67 bit for $x \geq 1$ and tends to zero as $x \to \infty$.

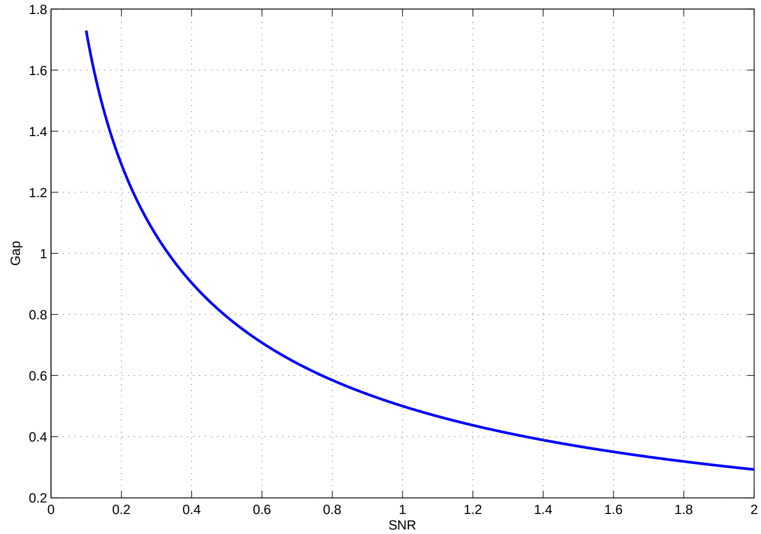

Figure 2. The gap between the outer bound and the achievable rate, given by (45).

Table I
THE GAP BETWEEN THE OUTER BOUND AND THE ACHIEVABLE RATE.

SNR	.1	.5	1	10	20
Gap (bits)	1.79	0.938	0.661	0.1257	0.0673

Note that, we can theorem (3) and theorem (5) to obtain the following achievable regions for other conditions over both noise variances.

Corollary 5. *If* $N_1 \leq \sqrt{a_{12}P_2P_1} - a_{12}P_2$ *and* $N_2 \geq \sqrt{a_{21}P_2P_1} - \min(a_{21}P_1, P_2)$ *for* $P_1 \neq a_{12}P_2$*,* $a_{21}P_1 \neq P_2$ *and* $P_1, P_2 \geq 1$*, then the following region is achievable for ASD-GIC:*

$$R_1 + R_2 \leq \min\left(\frac{1}{2}\log\left(1 + \frac{a_{12}P_2}{N_1}\right), u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_2 + a_{21}P_1 + N_2}{2N_2 + \left(\sqrt{P_2} - \sqrt{a_{21}}P_1\right)^2}\right)\right]^+\right\}\right), \tag{46}$$

where the upper convex envelope is with respect to P_1 *and* P_2*.*

Proof: By using (26) and (41), the proof is straightforward. ■

Corollary 6. *If* $N_1 \geq \sqrt{a_{12}P_2P_1} - \min(a_{12}P_2, P_1)$ *and* $N_2 \leq \sqrt{a_{21}P_2P_1} - a_{21}P_1$ *for* $P_1 \neq a_{12}P_2$*,* $a_{21}P_1 \neq P_2$ *and* $P_1, P_2 \geq 1$*, then the following region is achievable for ASD-GIC:*

$$R_1 + R_2 \leq \min\left(u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right\}, \frac{1}{2}\log\left(1 + \frac{a_{21}P_1}{N_2}\right)\right), \tag{47}$$

where the upper convex envelope is with respect to P_1 *and* P_2*.*

Proof: Using (27) and (40), the proof is straightforward. ■

V. Conclusion

In this paper, an additive state-dependent Gaussian interference channel (ASD-GIC) is considered. We assume that the state power goes to infinity. We establish four achievable rate regions by using lattice coding scheme. Dependent on noise variances, we reach to capacity or to a constant gap in the symmetric model.

Appendix

Appendix 1

Proposition 7. *For both MACs in ASD-GIC, in the limit of high SNR, where* $SNR_1 = \frac{P_1}{N_1} \gg 1$ *and* $SNR_2 = \frac{P_2}{N_2} \gg 1$*, the achievable sum-rate can be upper bounded by*

$$R_1 + R_2 \leq \left[h(S_1 + S_2) - h(S_1) - h(S_2) + \frac{1}{2}\log\left(\frac{2\pi e P_1 P_2}{N_1}\right)\right]^+. \tag{48}$$

Proof: We consider MAC 1. As for Costa dirty paper coding, we choose auxiliary random variables U_1 and U_2 as

$$\begin{aligned} U_1 &= X_1 + \alpha_1 S_1, \\ U_2 &= X_2 + \alpha_2 S_2, \end{aligned}$$

where $\alpha_1 = \frac{P_1}{P_1+N_1}$ and $\alpha_2 = \frac{a_{12}P_2}{a_{12}P_2+N_1}$. In the limit of high SNR, where $SNR_1 = \frac{P_1}{N_1} \gg 1$ and $SNR_2 = \frac{P_2}{N_2} \gg 1$, we have $\alpha_1 \approx 1, \alpha_2 \approx 1$. Thus, $U_1 = X_1 + S_1$, and $U_2 = X_2 + S_2$. Now, by using these auxiliary random variables in sum-rate provided in (15), we have

$$\begin{aligned} R_1 + R_2 &= \left[I\left(U_1, U_2; Y_1\right) - I(U_1; S_1) - I(U_2; S_2)\right]^+, \\ &= \left[h\left(Y_1\right) - h(Y_1|U_1, U_2) - h(U_1) + h(X_1) - h(U_2) + h(X_2)\right]^+, \\ &\leq \left[h\left(Y_1\right) - h(Z_1) - h(S_1) + h(X_1) - h(S_2) + h(X_2)\right]^+, \\ &= \left[h\left(Y_1\right) - h(S_1) - h(S_2) + \Gamma\right]^+, \\ &\leq \left[h\left(S_1 + S_2\right) - h(S_1) - h(S_2) + \Gamma\right]^+, \end{aligned}$$

where $[x]^+ = \max\{x, 0\}$ and $\Gamma = \frac{1}{2}\log\left(\frac{2\pi e P_1 P_2}{N_1}\right)$. For MAC 2, we can obtain similar result. ■

Now, by evaluating the upper bound in (48) for $Q_i \to \infty$, we get

$$\lim_{Q_i \to \infty} \left[h\left(S_1 + S_2\right) - h(S_1) - h(S_2) + \Gamma\right]^+ = \lim_{Q_i \to \infty} \left[\frac{1}{2}\log\left(\frac{Q_1 + Q_2}{Q_1 Q_2}\right) + \Gamma\right]^+ \to 0.$$

Thus, for $Q_i \to \infty$, we cannot reach any positive rate by random binning scheme.

Appendix 2

Here, we obtain the following corner point:

$$(R_1, R_2) = \left(\left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+, 0\right).$$

We assume that Λ_1 and Λ_2 are two lattices, which are good for quantization, with second moments P_1 and P_2, respectively.

First, we consider $P_1 \leq \frac{(a_{12}P_2+N_1)^2}{a_{12}P_2}$. For this case, we assume that $\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\Lambda_2 = \Lambda_1$. The encoders send

$$\boldsymbol{X}_1 = [\boldsymbol{V}_1 - \alpha_1 \boldsymbol{S}_1 + \boldsymbol{D}_1] \bmod \Lambda_1,$$

$$\boldsymbol{X}_2 \quad = \quad \left[-\alpha_2 \boldsymbol{S}_2 + \boldsymbol{D}_2\right] \mod \Lambda_2.$$

At the receiver of user 1, based on the channel output, given by

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \sqrt{a_{12}}\boldsymbol{X}_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1,$$

the following operation is performed:

$$\begin{aligned}
\boldsymbol{Y}_{d1} &= \left[\alpha_1 \boldsymbol{Y}_1 - \sqrt{a_{12}}\frac{\alpha_1}{\alpha_2}\boldsymbol{D}_2 - \boldsymbol{D}_1\right] \mod \Lambda_1, \\
&= \left[\alpha_1 \left(\boldsymbol{X}_1 + \sqrt{a_{12}}\left[-\alpha_2 \boldsymbol{S}_2 + \boldsymbol{D}_2\right] \mod \Lambda_2 + \boldsymbol{S}_1 + \sqrt{a_{12}}\boldsymbol{S}_2 + \boldsymbol{Z}_1\right) - \sqrt{a_{12}}\frac{\alpha_1}{\alpha_2}\boldsymbol{D}_2 - \boldsymbol{D}_1\right] \mod \Lambda_1, \\
&= \Big[\boldsymbol{V}_1 + \alpha_1\left(\boldsymbol{X}_1 + \boldsymbol{Z}_1\right) - \left(\boldsymbol{V}_1 - \alpha_1 \boldsymbol{S}_1 + \boldsymbol{D}_1\right) - \sqrt{a_{12}}\left(1-\alpha_2\right)\frac{\alpha_1}{\alpha_2}\left(-\alpha_2 \boldsymbol{S}_2 + \boldsymbol{D}_2\right) \\
&\quad - \sqrt{a_{12}}\alpha_1 \mathcal{Q}_{\Lambda_2}\left(-\alpha_2 \boldsymbol{S}_2 + \boldsymbol{D}_2\right)\Big] \mod \Lambda_1, \\
&= \left[\boldsymbol{V}_1 + \left(\alpha_1 - 1\right)\boldsymbol{X}_1 - \sqrt{a_{12}}\left(1-\alpha_2\right)\frac{\alpha_1}{\alpha_2}\boldsymbol{X}_2 + \alpha_1 \boldsymbol{Z}_1 - \sqrt{a_{12}}\frac{\alpha_1}{\alpha_2}\mathcal{Q}_{\Lambda_2}\left(-\alpha_2 \boldsymbol{S}_2 + \boldsymbol{D}_2\right)\right] \mod \Lambda_1, \qquad (49) \\
&= \left[\boldsymbol{V}_1 + \left(\alpha_1 - 1\right)\boldsymbol{X}_1 - \sqrt{a_{12}}\left(1-\alpha_2\right)\frac{\alpha_1}{\alpha_2}\boldsymbol{X}_2 + \alpha_1 \boldsymbol{Z}_1\right] \mod \Lambda_1, \qquad (50) \\
&= \left[\boldsymbol{V}_1 + \boldsymbol{Z}_{eff}\right] \mod \Lambda_1,
\end{aligned}$$

where

$$\boldsymbol{Z}_{eff} = \left[\left(\alpha_1 - 1\right)\boldsymbol{X}_1 - \sqrt{a_{12}}\left(1-\alpha_2\right)\frac{\alpha_1}{\alpha_2}\boldsymbol{X}_2 + \alpha_1 \boldsymbol{Z}_1\right] \mod \Lambda_1.$$

(49) is based on distributive law and (50) follows from $\frac{\alpha_1}{\alpha_2}\sqrt{a_{12}}\Lambda_2 = \Lambda_1$, we have that $\sqrt{a_{12}}\frac{\alpha_1}{\alpha_2}\mathcal{Q}_{\Lambda_2}\left(\alpha_2 \boldsymbol{S}_2 + \boldsymbol{D}_2\right) \in \Lambda_1$, i.e., the interference signal is aligned with Λ_1. To calculate the rate R_1, it is assumed that $\boldsymbol{V}_1 \sim$ Unif$\left(\mathcal{V}_1\right)$. We have

$$\begin{aligned}
R_1 &= \frac{1}{n} I\left(\boldsymbol{V}_1; \boldsymbol{Y}_{d1}\right), \\
&= \frac{1}{n}\left\{h(\boldsymbol{Y}_{d1}) - h(\boldsymbol{Y}_{d1}|\boldsymbol{V}_1)\right\} \\
&= \frac{1}{2}\log\left(\frac{P_1}{G\left(\Lambda_1\right)}\right) - \frac{1}{n}h\left(\left[\left(\alpha_1 - 1\right)\boldsymbol{X}_1 - \sqrt{a_{12}}\left(1-\alpha_2\right)\frac{\alpha_1}{\alpha_2}\boldsymbol{X}_2 + \alpha_1 \boldsymbol{Z}_1\right] \mod \Lambda_1\right), \qquad (51) \\
&\geq \frac{1}{2}\log\left(\frac{P_1}{\left(\alpha_1 - 1\right)^2 P_1 + \left(\left(1-\alpha_2\right)\frac{\alpha_1}{\alpha_2}\right)^2 a_{12}P_2 + \alpha_1^2 N_1}\right) - \frac{1}{2}\log\left(2\pi e G\left(\Lambda_1\right)\right), \qquad (52)
\end{aligned}$$

where (51) follows from the fact that $\sqrt{a_{12}}\boldsymbol{V}_2$ is uniform over $\sqrt{a_{12}}\mathcal{V}_2$, thus so $\boldsymbol{Y}_{d1}$ is uniform over $\sqrt{a_{12}}\mathcal{V}_2$ (crypto lemma). (52) follows from the fact that modulo operation reduces the second moment and Gaussian distribution maximizes differential entropy for a fixed second moment. Now, by considering $\left(\frac{\alpha_1}{\alpha_2}\right)^2 a_{12}P_2 = P_1$, we find the optimal α when the lattice dimension goes to infinity such that the MSE of the effective noise $\boldsymbol{Z}_{\text{eff}}$ is minimized. Hence,

$$\alpha_{1,\text{MMSE}} = \frac{\sqrt{P_1}\left(\sqrt{P_1} + \sqrt{a_{12}P_2}\right)}{P_1 + a_{12}P_2 + N_1}.$$

With this α, we get that the following achievable rate:

$$R_1 \leq \left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+. \qquad (53)$$

Thus, if $P_1 \leq \frac{\left(a_{12}P_2 + N_1\right)^2}{a_{12}P_2}$, we can achieve the following corner point:

$$(R_1,R_2)=\left(u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_1+a_{12}P_2+N_1}{2N_1+\left(\sqrt{P_1}-\sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right\},0\right) \tag{54}$$

Now, we consider $a_{12}P_2\le\frac{(P_1+N_1)^2}{P_1}$. For this case, we assume that $\Lambda_3=\frac{\alpha_2}{\alpha_1}\Lambda_1$, where $\Lambda_3=\sqrt{a_{12}}\Lambda_2$. The encoders send

$$\begin{aligned}\boldsymbol{X}_1 &= [\boldsymbol{V}_1-\alpha_1\boldsymbol{S}_1+\boldsymbol{D}_1] \bmod \Lambda_1,\\ \boldsymbol{X}_2 &= [-\alpha_2\boldsymbol{S}_2+\boldsymbol{D}_2] \bmod \Lambda_2.\end{aligned}$$

At the receiver of user 1, based on the channel output given by

$$\boldsymbol{Y}_1=\boldsymbol{X}_1+\sqrt{a_{12}}\boldsymbol{X}_2+\boldsymbol{S}_1+\sqrt{a_{12}}\boldsymbol{S}_2+\boldsymbol{Z}_1,$$

the following operation is performed:

$$\begin{aligned}\boldsymbol{Y}_{d1} &= \left[\alpha_2\boldsymbol{Y}_1-\sqrt{a_{12}}\boldsymbol{D}_2-\frac{\alpha_2}{\alpha_1}\boldsymbol{D}_1\right] \bmod \Lambda_3,\\
&= \left[\alpha_2\left([\boldsymbol{V}_1-\alpha_1\boldsymbol{S}_1+\boldsymbol{D}_1] \bmod \Lambda_1+\sqrt{a_{12}}\boldsymbol{X}_2+\boldsymbol{S}_1+\sqrt{a_{12}}\boldsymbol{S}_2+\boldsymbol{Z}_1\right)-\sqrt{a_{12}}\boldsymbol{D}_2-\frac{\alpha_2}{\alpha_1}\boldsymbol{D}_1\right] \bmod \Lambda_3,\\
&= \left[\frac{\alpha_2}{\alpha_1}\boldsymbol{V}_1+\alpha_2\left(\sqrt{a_{12}}\boldsymbol{X}_2+\boldsymbol{Z}_1\right)-\sqrt{a_{12}}\left(-\alpha_2\boldsymbol{S}_2+\boldsymbol{D}_2\right)-(1-\alpha_1)\frac{\alpha_2}{\alpha_1}\left(\boldsymbol{V}_1-\alpha_1\boldsymbol{S}_1+\boldsymbol{D}_1\right)\right.\\
&\quad \left.-\alpha_2\mathcal{Q}_{\Lambda_1}\left(\boldsymbol{V}_1-\alpha_1\boldsymbol{S}_1+\boldsymbol{D}_1\right)\right] \bmod \Lambda_3,\\
&= \left[\frac{\alpha_2}{\alpha_1}\boldsymbol{V}_1+\sqrt{a_{12}}\left(\alpha_2-1\right)\boldsymbol{X}_2-(1-\alpha_1)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1+\alpha_2\boldsymbol{Z}_1-\frac{\alpha_2}{\alpha_1}\mathcal{Q}_{\Lambda_1}\left(\boldsymbol{V}_1-\alpha_1\boldsymbol{S}_1+\boldsymbol{D}_1\right)\right] \bmod \Lambda_3, && (55)\\
&= \left[\frac{\alpha_2}{\alpha_1}\boldsymbol{V}_1+\sqrt{a_{12}}\left(\alpha_2-1\right)\boldsymbol{X}_2-(1-\alpha_1)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1+\alpha_2\boldsymbol{Z}_1\right] \bmod \Lambda_3, && (56)\\
&= \left[\frac{\alpha_2}{\alpha_1}\boldsymbol{V}_1+\boldsymbol{Z}_{eff}\right] \bmod \Lambda_3,\end{aligned}$$

where

$$\boldsymbol{Z}_{eff}=\left[\sqrt{a_{12}}\left(\alpha_2-1\right)\boldsymbol{X}_2-(1-\alpha_1)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1+\alpha_2\boldsymbol{Z}_1\right] \bmod \Lambda_3.$$

(55) is based on distributive law and (56) follows from $\frac{\alpha_2}{\alpha_1}\Lambda_1=\Lambda_3$, we have that $\frac{\alpha_2}{\alpha_1}\mathcal{Q}_{\Lambda_1}\left(\alpha_2\boldsymbol{S}_2+\boldsymbol{D}_2\right)\in\Lambda_3$, i.e., the interference signal is aligned with Λ_3. Hence, the element disappears after the modulo operation. To calculate rate R_1, it is assumed that $\boldsymbol{V}_1\sim$ Unif$\left(\mathcal{V}_1\right)$. We have

$$\begin{aligned}R_1 &= \frac{1}{n}I\left(\boldsymbol{V}_1;\boldsymbol{Y}_{d1}\right),\\
&= \frac{1}{n}\left\{h(\boldsymbol{Y}_{d1})-h(\boldsymbol{Y}_{d1}|\boldsymbol{V}_1)\right\}\\
&= \frac{1}{2}\log\left(\frac{a_{12}P_2}{G\left(\Lambda_3\right)}\right)-\frac{1}{n}h\left(\left[\sqrt{a_{12}}\left(\alpha_2-1\right)\boldsymbol{X}_2-(1-\alpha_1)\frac{\alpha_2}{\alpha_1}\boldsymbol{X}_1+\alpha_2\boldsymbol{Z}_1\right] \bmod \Lambda_3\right), && (57)\\
&\ge \frac{1}{2}\log\left(\frac{a_{12}P_2}{\left(\alpha_2-1\right)^2a_{12}P_1+\left((1-\alpha_1)\frac{\alpha_2}{\alpha_1}\right)^2P_1+\alpha_2^2N_1}\right)-\frac{1}{2}\log\left(2\pi eG\left(\Lambda_3\right)\right), && (58)\end{aligned}$$

Since $\frac{\alpha_2}{\alpha_1}\boldsymbol{V}_1$ is uniform over $\frac{\alpha_2}{\alpha_1}\mathcal{V}_2$, $\boldsymbol{Y}_{d1}$ is also uniform over $\frac{\alpha_2}{\alpha_1}\mathcal{V}_2$ (crypto lemma), thus (57) is correct. (58) follows from the fact that modulo operation reduces the second moment and Gaussian distribution maximizes differential entropy for a fixed second moment. Now, by considering $\left(\frac{\alpha_2}{\alpha_1}\right)^2P_1=a_{12}P_2$, and the MMSE value of α, which minimizes the MSE of the effective noise, $\boldsymbol{Z}_{\text{eff}}$,

$$\alpha_{2,\text{MMSE}}=\frac{\sqrt{a_{12}P_2}\left(\sqrt{P_1}+\sqrt{a_{12}P_2}\right)}{P_1+a_{12}P_2+N_1}.$$

we get the following achievable rate:

$$R_1 \leq \left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+. \tag{59}$$

Thus, if $a_{12}P_2 \leq \frac{(P_1+N_1)^2}{P_1}$, then we can achieve the following corner point:

$$(R_1, R_2) = \left(u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right\}, 0\right) \tag{60}$$

Now, by combining (54) and (60), we get the following corner point

$$(R_1, R_2) = \left(u.c.e\left\{\left[\frac{1}{2}\log\left(\frac{P_1 + a_{12}P_2 + N_1}{2N_1 + \left(\sqrt{P_1} - \sqrt{a_{12}P_2}\right)^2}\right)\right]^+\right\}, 0\right),$$

if

$$N_1 \geq \sqrt{a_{12}P_1P_2} - \min\left(a_{12}P_2, P_1\right).$$

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